

State-feedback Stabilization of Markov Jump Linear Systems with Randomly Observed Markov States

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Abstract—In this paper we study the state-feedback stabilization of a discrete-time Markov jump linear system when the observation of the Markov chain of the system, called the Markov state, is time-randomized by another Markov chain. Embedding the Markov state into an extended Markov chain, we transform the given system with time-randomized observations to another one having the enlarged Markov-state space but with so-called cluster observations of Markov states. Based on this transformation we propose linear matrix inequalities for designing stabilizing state-feedback gains for the original Markov jump linear systems. The proposed method can treat both periodic observations and many of renewal-type observations in a unified manner, which are studied in the literature using different approaches. A numerical example is provided to demonstrate the obtained result.

I. INTRODUCTION

Markov jump linear systems is a class of switched linear systems whose switching is governed by a time-homogeneous Markov process, called the Markov state, and have been attracting continuing attention due to its simplicity as well as its ability of modeling systems in application such as robotic systems [1], [2], economy [3], [4], and networked systems [5]. It is known that, under the assumption that controllers can observe the Markov state at any time instants, we can perform standard types of controller synthesis for Markov jump linear systems such as state-feedback stabilization, quadratic optimal control, H^2 optimal control, and H^∞ optimal control (see, e.g., the monograph [6]).

However it is often not realistic to assume that controllers *always* have an access to the Markov state and this fact has been motivating the investigation of the effect of limited or uncertain observations of the Markov state. For example, the authors in [7] study the stabilization and H^2 -control of discrete-time Markov jump linear systems when the Markov-state space is partitioned into subsets, called clusters, and an observation of the Markov state only tells us to which cluster the Markov-state belongs. Similar studies in the continuous-time settings can be found in [8], [9]. Under an extreme situation when the Markov-state space has only one cluster, i.e., when one cannot observe the Markov state, Vargas et al. [10] investigate quadratic optimal control problems.

Another but not the only source of uncertainty comes from the randomness of the time instants at which one can observe the Markov states. For the case when observation

times follow a renewal process, the authors in [11] design almost-surely stabilizing state-feedback controllers whose gains are reset whenever an observation is performed. For the special case when observations are performed periodically, the same authors [12], [13] derive stabilizing (in the mean square sense) state-feedback controllers using Lyapunov-like functions. We here remark that other various methods such as, for example, adaptive strategies [14] could be used to study this type of problems, though we do not give a detailed survey of the field in this paper.

In this paper we propose a unified method for designing stabilizing state-feedback gains for a discrete-time Markov jump linear system when the time instants at which a controller performs an observation of the Markov state, called an observation process, is time-randomized by another Markov chain. This Markov chain can be used to model various types of observation processes including periodic observations [12], [13] and observations following a renewal process [11]. By embedding the original Markov-state to another one, we transform a Markov jump linear system with time-randomized observations to another one with clustered observations, for which we apply the result in [7] and derive linear matrix inequalities for finding stabilizing state-feedback gains.

This paper is organized as follows. After preparing the notations used in this paper, in Section II we give a brief overview of Markov jump linear systems and their stabilization. Then in Section III we formulate the stabilization problem with time-random observation of the Markov state. After showing an embedding of the Markov state to another Markov chain in Section IV, we in Section V derive linear matrix inequalities for the design of feedback gains.

A. Mathematical Preliminaries

The notation used in this paper is standard. Let \mathbb{N} denote the set of nonnegative integers. Let \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote the vector spaces of real n -vectors and $n \times m$ matrices, respectively. By $\|\cdot\|$ we denote the Euclidean norm on \mathbb{R}^n . $\mathcal{P}(\cdot)$ will be used to denote the probability of an event. The probability of an event conditional on an event \mathcal{E} is denoted by $\mathcal{P}(\cdot | \mathcal{E})$. Expectations are denoted by $E[\cdot]$. Characteristic functions are denoted by $\mathbb{1}(\cdot)$. For a positive integer N we define the set $[N] = \{1, \dots, N\}$. For a positive integer T define $[k]_T$ as the unique integer in $[T]$ such that $k - [k]_T$ is an integer multiple of T . When a real symmetric matrix A is positive definite we write $A > 0$.

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II. MARKOV JUMP LINEAR SYSTEMS AND STABILIZATION

The aim of this section is to give a brief overview of Markov jump linear systems in discrete-time [6] and also recall some basic definitions of their stability and stabilizability.

Let n, m , and N be positive integers. Let $A_1, \dots, A_N \in \mathbb{R}^{n \times n}$ and $B_1, \dots, B_N \in \mathbb{R}^{m \times n}$. Also let $r = \{r(k)\}_{k=0}^\infty$ be a time-homogeneous Markov chain taking its values in $\mathfrak{X} = [N]$ and having the transition probability matrix $P \in \mathbb{R}^{N \times N}$. We call the stochastic difference equation

$$\Sigma: x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k) \quad (1)$$

a Markov jump linear system [6]. We call \mathfrak{X} the *Markov-state space* of Σ . Both the initial state $x(0) = x_0 \in \mathbb{R}^n$ and the initial Markov state $r(0) = r_0 \in \mathfrak{X}$ are assumed to be constants. The (internal) mean square stability of Σ is defined in the following standard way.

Definition 2.1: Σ is said to be *mean square stable* if there exist $C > 0$ and $\varepsilon > 0$ such that the solution x of (1) satisfies

$$E[\|x(k)\|^2] < C\varepsilon^k \|x_0\|^2 \quad (2)$$

for all x_0 and r_0 , provided $u = 0$.

In this paper we mainly discuss the stabilization of Σ via state-feedback controllers. If one assumes that the controller has an exact access to the Markov state $r(k)$ at each time $k \geq 0$, then we can consider the following mode-dependent controller of the form

$$u(k) = K_{r(k)}x(k) \quad (3)$$

where $K_1, \dots, K_N \in \mathbb{R}^{m \times n}$. We say that the state-feedback controller (3) *stabilizes* Σ if the following Markov jump linear system without input

$$x(k+1) = (A_{r(k)} + B_{r(k)}K_{r(k)})x(k)$$

is mean square stable.

Another scenario, which is closely related to the current paper, is the stabilization with so-called cluster observations of Markov states (see [7]). Following the notation in [7], we assume that the Markov state space \mathfrak{X} is decomposed as $\mathfrak{X} = \mathfrak{X}_h \times \mathfrak{X}_o$, where \mathfrak{X}_h and \mathfrak{X}_o are sets. Thus each $i \in \mathfrak{X}$ can be represented as $i = (i_h, j_o)$ by some $i_h \in \mathfrak{X}_h$ and $j_o \in \mathfrak{X}_o$. The set \mathfrak{X}_h (\mathfrak{X}_o) represents the unobservable (observable, respectively) part of the Markov state space \mathfrak{X} . Let us define the projection $\pi_o: \mathfrak{X} \rightarrow \mathfrak{X}_o$ by $\pi_o(i_h, j_o) = j_o$. Then, the state-feedback controller that can observe only the observable part of the Markov state must take the form

$$u(k) = K_{\pi_o(r(k))}x(k), \quad (4)$$

where $K_j \in \mathbb{R}^{m \times n}$ for each $j \in \mathfrak{X}_o$. We say that this feedback controller stabilizes Σ if the solution x of the closed loop equation $x(k+1) = (A_{r(k)} + B_{r(k)}K_{\pi_o(r(k))})x(k)$ satisfies the condition in Definition 2.1.

The following proposition [7, Theorem 6] gives linear matrix inequalities whose solutions yield stabilizing feedback gains for feedback control (4) with clustered observations.

In order to state the proposition, for $i \in \mathfrak{X}$ and a family of matrices $\{R_i\}_{i \in \mathfrak{X}} \subset \mathbb{R}^{n \times n}$ we define the matrix $\mathcal{D}_i(R) \in \mathbb{R}^{n \times n}$ by $\mathcal{D}_i(R) = \sum_{j=1}^N p_{ji}R_j$.

Proposition 2.2 ([7, Theorem 6]): Assume that the matrices $R_i \in \mathbb{R}^{n \times n}$, $G_j \in \mathbb{R}^{m \times n}$, and $F_j \in \mathbb{R}^{m \times n}$ ($i \in \mathfrak{X}_j$, $j \in \mathfrak{X}_o$) satisfy the matrix linear inequalities

$$\begin{bmatrix} R_i & A_i G_j + B_i F_j \\ G_j^\top A_i^\top + F_j^\top B_i^\top & G_j + G_j^\top - \mathcal{D}_i(R) \end{bmatrix} > 0$$

for all $i \in \{j\} \times \mathfrak{X}_o$ and $j \in \mathfrak{X}_o$. Define $K_j = F_j G_j^{-1}$ for each $j \in \mathfrak{X}_o$. Then the feedback controller (4) stabilizes Σ .

III. RANDOM OBSERVATION PROCESSES INDUCED BY MARKOV CHAINS

The aim of this section is to state the stabilization problem with time-randomly observed Markov states. We in particular introduce a novel class of random observation processes induced by Markov chains. In particular the class contains observation processes that are not renewal processes. Let us begin with the next general definition.

Definition 3.1: An \mathbb{N} -valued increasing stochastic process $t = \{t_i\}_{i=0}^\infty$ is called an *observation process*.

Observation processes will be used to model the times at which a controller can access the Markov state. Given an observation process t , define the stochastic process $\tau = \{\tau(k)\}_{k=0}^\infty$ by

$$\tau(k) = \begin{cases} \max\{t_i : t_i \leq k\} & k \geq \max(0, t_0) \\ \tau_0 & \text{otherwise} \end{cases}$$

where $\tau_0 < 0$ is an arbitrary integer. This $\tau(k)$ represents, for each time k , the most recent time the Markov state was observed. In particular we have $\tau(t_i) = t_i$ for every $i \geq 0$. Notice that we augment the process with the arbitrary negative integer τ_0 when $k < \max(0, t_0)$ because, before the time $k = t_0$, no observation is performed yet.

We then define another stochastic process $\sigma = \{\sigma(k)\}_{k=0}^\infty$ taking its values in $[N]$ by

$$\sigma(k) = \begin{cases} r(\tau(k)) & k \geq \max(0, t_0) \\ \sigma_0 & \text{otherwise} \end{cases}$$

where $\sigma_0 \in [N]$ is arbitrary. The process σ represents the most-updated information of the Markov state that is available for a controller. We again notice that, by the same reason as above, the process σ is augmented by an arbitrary σ_0 before the time $k = \max(0, t_0)$, i.e., before the first observation is performed. See Fig. 1 for an illustration.

In this paper we assume that the controller has an access to, at each time $k \geq 0$, the state variable $x(k)$, the most recent observation $\sigma(k)$ of the Markov state r , and $k - \tau(k)$, which is the time elapsed since the last observation. Then we construct the state-feedback controller of the form

$$u(k) = K_{\sigma(k), [k+1-\tau(k)]_T} x(k), \quad (5)$$

where $K_{\gamma, \delta} \in \mathbb{R}^{m \times n}$ for each $\gamma \in [N]$ and $\delta \in [T]$. The first argument $\sigma(k)$ in (5) allows the gain to be reset whenever a controller performs an observation of the Markov state as

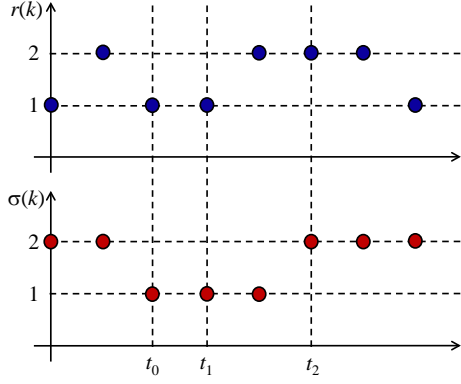


Fig. 1. Markov state r and its observed version σ . Until the first observation time t_0 , the observed version is temporarily set to $\sigma_0 = 2$.

in [11]–[13]. The second argument allows the controller to change feedback gains between two consecutive observations rather than keeping them to be constant, which can enhance the performance of the controller [13]. The reason for taking the operator $[\cdot]_T$ in the second argument of K is that, otherwise, we have to design infinitely many matrices $K_{\gamma, \delta}$ where δ could be any nonnegative numbers. Taking the operator $[\cdot]_T$ forces δ to be in the finite set $[T]$, which turns out to make our stabilization problem solvable in finite time.

Combining (1) and (5) we obtain the closed loop equation

$$\Sigma_K : x(k+1) = (A_{r(k)} + B_{r(k)} K_{\sigma(k), [k+1-\tau(k)]_T}) x(k). \quad (6)$$

Extending Definition 2.1, we define the mean square stability of the system Σ_K as follows.

Definition 3.2: Let \mathcal{T} be a set of observation processes. We say that the pair (Σ_K, \mathcal{T}) is *mean square stable* if there exist $C > 0$ and $\varepsilon \in [0, 1)$ such that the solution x of (6) satisfies (2) for all $x_0 \in \mathbb{R}^n$, $r_0 \in [N]$, $\tau_0 < 0$, $\sigma_0 \in [N]$, and $t \in \mathcal{T}$. The feedback control (5) is said to stabilize (Σ, \mathcal{T}) if (Σ_K, \mathcal{T}) is mean square stable.

A. Observation Process Induced by Markov Chains

In this paper we deal with a class of observation processes induced by time-homogeneous Markov chains. In order to introduce the class, we first need to define observation processes induced by deterministic sequences. Let $s: \mathbb{N} \rightarrow [M]$ be an arbitrary sequence and let Λ be a subset of $[M]$. Assume that s intersects with M infinitely many times, namely, that the set $\{k \in \mathbb{N} : s(k) \in \Lambda\}$ is infinite. Then define the infinite sequence $t_\Lambda(s)$ as the one obtained by increasingly ordering the numbers in the infinite set $\{k \in \mathbb{N} : s(k) \in \Lambda\}$. Thus, the sequence $t_\Lambda(s)$ consists of the times k at which the sequence s intersects with Λ . For example, the observation time instants t_0, t_1 , and t_2 shown in Fig. 1 are induced by the sequence s shown in Fig. 2 with $\Lambda = \{2\}$.

Then we extend the above definition to Markov chains as follows. Let s be a time-homogeneous Markov chain taking its values in $[M]$ and let $\Lambda \subset [M]$ be a nonempty set that is recurrent with respect to the Markov chain s . We define a family of observation processes $\mathcal{T}_{s, \Lambda}$ by

$$\mathcal{T}_{s, \Lambda} = \{t_\Lambda(s(\cdot; s_0))\}_{s_0=1}^M, \quad (7)$$

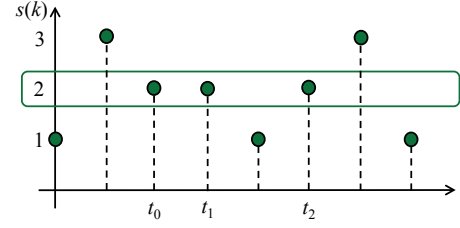


Fig. 2. A sequence s and the induced observation time instants t . An observation occurs whenever s enters the set $\Lambda = \{2\}$.

where $s(\cdot; s_0)$ denotes the Markov chain s when its initial state equals s_0 . Notice that, since Λ is recurrent, s intersects with Λ infinitely many times with probability one and thereby $\mathcal{T}_{s, \Lambda}$ is well defined. The following examples illustrate that the family $\mathcal{T}_{s, \Lambda}$ can express various types of observation processes.

Example 3.3 (Periodic observation with failures): Let τ be a positive integer. Let $\Lambda = \{1\}$ and the transition probability matrix of s be

$$\begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \end{bmatrix} \in \mathbb{R}^{(\tau+1) \times (\tau+1)},$$

where zero entries are omitted. Then we can see that, if $t \in \mathcal{T}_{s, \Lambda}$, then the difference $t_{i+1} - t_i$ of observation times independently follow the distribution μ on \mathbb{N} that is concentrated on the set $\{\tau, 2\tau, 3\tau, \dots\}$ and satisfies $\mu(\{\tau k\}) = (1-p)^{k-1}p$ for every $k \geq 1$. In other words, this observation process expresses the observation of every τ time units with the probability of failure $1-p$ at each observation. In particular, if $p = 1$, then the observation process gives the observation with period τ , which is considered in [13].

Example 3.4 (Renewal processes): Let μ be an arbitrary distribution on $\mathbb{N} \setminus \{0\}$ having finite support. Then there exist $\tau > 0$ and $\{p_k\}_{k=1}^\tau$ such that $\sum_{k=1}^\tau p_k = 1$ and $\mu(\{k\}) = p_k$ for every $1 \leq k \leq \tau$. Define the positive integers $\tilde{p}_1, \dots, \tilde{p}_\tau$ recursively by $\tilde{p}_k = p_k / \prod_{\ell=1}^{k-1} (1 - \tilde{p}_\ell)$ for $k = 1, \dots, \tau$. Let s be the Markov chain having the transition probability matrix

$$\begin{bmatrix} \tilde{p}_1 & \text{---} & 1 - \tilde{p}_1 & \text{---} & \text{---} & \text{---} \\ \vdots & \text{---} & \text{---} & \ddots & \text{---} & \text{---} \\ \tilde{p}_{\tau-1} & \text{---} & \text{---} & \text{---} & 1 - \tilde{p}_{\tau-1} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \end{bmatrix} \in \mathbb{R}^{\tau \times \tau},$$

where zero entries are omitted again. Also let $\Lambda = \{1\}$. We can see that, if $t \in \mathcal{T}_{s, \Lambda}$, then difference $t_{i+1} - t_i$ of observation times independently follow the distribution μ and therefore t forms a renewal process.

Finally we present a simple example of observation processes $\mathcal{T}_{s, \Lambda}$ that are not renewal processes and therefore cannot be treated by the method in [11].

Example 3.5: Let the transition probability matrix of s be

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and also let $\Lambda = \{1, 3\}$. If $s_0 = 1$, then we have $\{t_i\}_{i=0}^\infty = \{0, 2, 3, 5, 6, 8, 9, \dots\}$, which cannot be a renewal process because the differences of two consecutive observation times equal $\{2, 1, 2, 1, 2, \dots\}$ almost surely.

Now we state the main problem studied in this paper.

Problem 3.6: Given a Markov jump linear system Σ and a family of observation processes $\mathcal{T}_{s,\Lambda}$ induced by a time-homogeneous Markov chain s , find feedback gains $K = \{K_{\gamma,\delta}\}_{\gamma \in [N], \delta \in [T]}$ that stabilize the pair $(\Sigma, \mathcal{T}_{s,\Lambda})$.

IV. EMBEDDING OF MARKOV STATES

The difficulty of solving Problem 3.6 is that the system Σ_K in (6) is no longer a standard Markov jump linear system and thus the techniques established in the literature [6] cannot be used to it. Also, by the generality of the observation processes $\mathcal{T}_{s,\Lambda}$ discussed in the previous section, we also cannot use the results recently proposed in [11]–[13]. The aim of this section is to show that we can embed the Markov state process r to another Markov chain, with which we can express Σ_K as a standard Markov jump linear system. This fact will be used in the next section to reduce Problem 3.6 to the stabilization of a Markov jump linear system with clustered observations.

Let r be the Markov state of the Markov jump linear system Σ . Let s be the time-homogeneous Markov chain that induces the family $\mathcal{T}_{s,\Lambda}$ of observation processes. We let $P = [p_{ij}]_{i,j} \in \mathbb{R}^{N \times N}$ and $Q = [q_{ij}]_{i,j} \in \mathbb{R}^{M \times M}$ denote the transition probability matrices of r and s , respectively. The next proposition is the main result of this section.

Proposition 4.1: Define

$$\tilde{\mathcal{X}} = [N] \times [M] \times [N] \times [T].$$

Then the $\tilde{\mathcal{X}}$ -valued stochastic process \bar{r} defined by

$$\bar{r}(k) = (r(k), s(k), \sigma(k), [k+1 - \tau(k)]_T), \quad k \geq 0 \quad (8)$$

is a time-homogeneous Markov chain. Moreover its transition probabilities are given by, for all $\chi = (\alpha, \beta, \gamma, \delta)$ and $\chi' = (\alpha', \beta', \gamma', \delta')$ in $\tilde{\mathcal{X}}$,

$$\begin{aligned} \mathcal{P}(\bar{r}(k+1) = \chi' \mid \bar{r}(k) = \chi) \\ = \begin{cases} \mathbb{1}(\alpha' = \gamma', \delta = 1) p_{\alpha,\alpha'} q_{\beta,\beta'} & \beta' \in \Lambda, \\ \mathbb{1}(\gamma' = \gamma, \delta' = [\delta + 1]_T) p_{\alpha,\alpha'} q_{\beta,\beta'} & \beta' \notin \Lambda. \end{cases} \quad (9) \end{aligned}$$

Proof: Let $k_0 \in \mathbb{N}$ and $k \geq k_0$ be arbitrary. Take arbitrary $\chi_i = (\alpha_i, \beta_i, \gamma_i, \delta_i) \in \tilde{\mathcal{X}}$ ($i = k_0, \dots, k+1$). For each i define the events $\mathcal{E}_i, \mathcal{F}_i$ by $\mathcal{E}_i = \{\bar{r}(i) = \chi_i, \dots, \bar{r}(k_0) = \chi_{k_0}\}$ and

$$\begin{aligned} \mathcal{F}_i &= \{\bar{r}(i) = \chi_i\} \\ &= \{r(i) = \alpha_i, s(i) = \beta_i, \sigma(i) = \gamma_i, [i+1 - \tau(i)]_T = \delta_i\}. \end{aligned}$$

Under the assumption that \mathcal{E}_k is not the null set, we need to evaluate the conditional probability

$$\mathcal{P}(\bar{r}(k+1) = \chi_{k+1} \mid \mathcal{E}_k) = \mathcal{P}(\mathcal{E}_{k+1}) / \mathcal{P}(\mathcal{E}_k). \quad (10)$$

Remark that this assumption implies that

$$\sigma(k) = \gamma_k, [k+1 - \tau(k)]_T = \delta_k, \quad (11)$$

because otherwise \mathcal{E}_k equals a null set.

First assume that $\beta_{k+1} \in \Lambda$. Then we have $s(k+1) \in \Lambda$ so that, by the definition of $\mathcal{T}_{s,\Lambda}$, an observation occurs at time $k+1$, i.e., we have $\tau(k+1) = k+1$ and $\sigma(k+1) = r(k+1)$. This implies that

$$\begin{aligned} \mathcal{F}_{k+1} &= \{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}, \\ &\quad \alpha_{k+1} = \gamma_{k+1}, [1]_T = \delta_{k+1}\}. \end{aligned}$$

Therefore, since $\mathcal{E}_{k+1} = \mathcal{E}_k \cap \mathcal{F}_{k+1}$,

$$\begin{aligned} \mathcal{E}_{k+1} &= \{\alpha_{k+1} = \gamma_{k+1}, \delta_{k+1} = 1\} \\ &\quad \cap \{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{E}_k \end{aligned} \quad (12)$$

and hence

$$\begin{aligned} \mathcal{P}(\mathcal{E}_{k+1}) &= \mathbb{1}(\alpha_{k+1} = \gamma_{k+1}, \delta_{k+1} = 1) \\ &\quad \mathcal{P}(\{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{E}_k). \end{aligned} \quad (13)$$

The probability appearing in the last term of this equation can be computed as

$$\begin{aligned} &\mathcal{P}(\{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{E}_k) \\ &= \mathcal{P}(\mathcal{E}_k) \mathcal{P}(r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1} \mid \mathcal{E}_k) \\ &= \mathcal{P}(\mathcal{E}_k) \mathcal{P}(r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1} \\ &\quad \mid r(k) = \alpha_k, s(k) = \beta_k) \\ &= \mathcal{P}(\mathcal{E}_k) p_{\alpha_k, \alpha_{k+1}} q_{\beta_k, \beta_{k+1}}, \end{aligned} \quad (14)$$

where we used the fact that both r and s are time-homogeneous Markov chains. Thus equations (10), (13), and (14) conclude that, for the case of $\beta_{k+1} \in \Lambda$,

$$\begin{aligned} \mathcal{P}(\bar{r}(k+1) = \chi_{k+1} \mid \mathcal{E}_k) \\ = \mathbb{1}(\alpha_{k+1} = \gamma_{k+1}, \delta_{k+1} = 1) p_{\alpha_k, \alpha_{k+1}} q_{\beta_k, \beta_{k+1}}. \end{aligned} \quad (15)$$

Then consider the case where $\beta_{k+1} \notin \Lambda$. In this case, Markov state r is not observed at time $k+1$ so that we have $\tau(k+1) = \tau(k)$ and $\sigma(k+1) = \sigma(k)$. Therefore, using equations (11), in the same way as we derived (12) we can show that

$$\begin{aligned} \mathcal{E}_{k+1} &= \{\gamma_{k+1} = \gamma_k, \delta_{k+1} = [\delta_k + 1]_T\} \\ &\quad \cap \{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{E}_k \end{aligned} \quad (16)$$

and hence

$$\begin{aligned} \mathcal{P}(\mathcal{E}_{k+1}) &= \mathbb{1}(\gamma_{k+1} = \gamma_k, \delta_{k+1} = [\delta_k + 1]_T) \\ &\quad \mathcal{P}(\{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{E}_k). \end{aligned}$$

Therefore, equations (10), (14), and (16) show that, for $\beta_{k+1} \notin \Lambda$,

$$\begin{aligned} \mathcal{P}(\bar{r}(k+1) = \chi_{k+1} \mid \mathcal{E}_k) \\ = \mathbb{1}(\gamma_k = \gamma_{k+1}, \delta_{k+1} = [\delta_k + 1]_T) p_{\alpha_k, \alpha_{k+1}} q_{\beta_k, \beta_{k+1}}. \end{aligned} \quad (17)$$

Since the probabilities (15) and (17) do not depend on k_0 , letting $k_0 = k$ and $k_0 = 0$ in (15) and (17) we obtain

$$\begin{aligned} \mathcal{P}(\bar{r}(k+1) = \chi_{k+1} \mid \bar{r}(k) = \chi_k, \dots, \bar{r}(0) = \chi_0) \\ = \mathcal{P}(\bar{r}(k+1) = \chi_{k+1} \mid \bar{r}(k) = \chi_k) \end{aligned}$$

for every $k \geq 0$. This shows that \bar{r} is a Markov chain since $\chi_0, \dots, \chi_{k+1} \in \tilde{\mathcal{X}}$ were arbitrarily taken. Moreover, since the probabilities (15) and (17) do not depend on k , we conclude that the Markov chain \bar{r} is time-homogeneous and its transition probabilities are actually given by (9). ■

Remark 4.2: It is observed in [13] that the process $\{r(k), \sigma(k)\}_{k \geq 0}$ itself is indeed a Markov chain but not time-homogeneous. Proposition 4.1 shows that, however, augmenting the third and fourth components in (8) enables us to construct a time-homogeneous Markov chain, which plays a crucial role in the next section.

V. DESIGNING STABILIZING FEEDBACK GAINS VIA LINEAR MATRIX INEQUALITIES

In this section we show a set of feedback gains K that stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$ can be found by solving a set of linear matrix inequalities. For the proof we will use the reduction of Σ_K , which is not necessarily a Markov jump linear system, to a Markov jump linear system with Markov state evolving in the same way as the Markov chain \bar{r} presented in the last section.

We define θ as the time-homogeneous Markov chain taking its values in $\tilde{\mathcal{X}}$ and having the same transition probability as \bar{r} , i.e., we assume that the transition probability $\bar{p}_{\chi,\chi'}$ of θ is given by

$$\bar{p}_{\chi,\chi'} = \begin{cases} \mathbb{1}(\alpha' = \gamma', \delta = 1) p_{\alpha,\alpha'} q_{\beta,\beta'} & \beta' \in \Lambda \\ \mathbb{1}(\gamma' = \gamma, \delta' = \lfloor \delta + 1 \rfloor_T) p_{\alpha,\alpha'} q_{\beta,\beta'} & \beta' \notin \Lambda \end{cases}$$

where $\chi' = (\alpha', \beta', \gamma', \delta')$ and $\chi = (\alpha, \beta, \gamma, \delta)$. The following lemma will be used to prove the main result of this paper.

Lemma 5.1: Let $r_0 \in [N]$, $s_0 \in [M]$, $\tau_0 < 0$, and $\sigma_0 \in [N]$ be arbitrary. Then there exists $\chi_0 \in \tilde{\mathcal{X}}$ such that

$$\theta(k; \chi_0) = \bar{r}(k; r_0, s_0, \sigma_0, \lfloor 1 - \tau_0 \rfloor_T), \quad (18)$$

where the arguments following the time k denote the initial conditions of θ and \bar{r} .

Now, for the Markov jump linear system Σ , we introduce another Markov jump linear system having θ as its Markov state as follows. We define the matrices $\bar{A}_\chi \in \mathbb{R}^{n \times n}$ and $\bar{B}_\chi \in \mathbb{R}^{n \times m}$ for each $\chi = (\alpha, \beta, \gamma, \delta) \in \tilde{\mathcal{X}}$ by $\bar{A}_\chi = A_\alpha$ and $\bar{B}_\chi = B_\alpha$. Then define the Markov jump linear system $\bar{\Sigma}$ by

$$\bar{\Sigma} : \bar{x}(k+1) = \bar{A}_{\theta(k)} \bar{x}(k) + \bar{B}_{\theta(k)} \bar{u}(k),$$

with the initial states $\bar{x}(0) = \bar{x}_0 \in \mathbb{R}^n$ and $\bar{\theta}(0) = \bar{\theta}_0 \in \tilde{\mathcal{X}}$. Let us also consider the following standard state-feedback controller

$$\bar{u}(k) = \bar{K}_{\theta(k)} \bar{x}(k) \quad (19)$$

where $\bar{K}_\chi \in \mathbb{R}^{m \times n}$ for each $\chi \in \tilde{\mathcal{X}}$. Then we obtain the closed loop equation

$$\bar{\Sigma}_{\bar{K}} : \bar{x}(k+1) = (\bar{A}_{\theta(k)} + \bar{B}_{\theta(k)} \bar{K}_{\theta(k)}) \bar{x}(k).$$

The next theorem is the first major result of this paper.

Theorem 5.2: Assume that $\{\bar{K}_\chi\}_{\chi \in \tilde{\mathcal{X}}} \subset \mathbb{R}^{m \times n}$ stabilizes $\bar{\Sigma}$ and satisfies

$$\bar{K}_{\alpha,\beta,\gamma,\delta} = \bar{K}_{\alpha',\beta',\gamma,\delta} \quad (20)$$

for all $\alpha, \alpha' \in [N]$, $\beta, \beta' \in [M]$, $\gamma \in [N]$, and $\delta \in [T]$. For each $\gamma \in [N]$ and $\delta \in [T]$ define $K_{\gamma,\delta}$ by

$$K_{\gamma,\delta} = \bar{K}_{1,1,\gamma,\delta}. \quad (21)$$

Then K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$.

Proof: Assume that $\{\bar{K}_\chi\}_{\chi \in \tilde{\mathcal{X}}} \subset \mathbb{R}^{m \times n}$ stabilizes $\bar{\Sigma}$ and satisfies (20). Then there exist $\bar{C} > 0$ and $\varepsilon \in [0, 1)$ such that the solution \bar{x} of $\bar{\Sigma}_{\bar{K}}$ satisfies $E[\|\bar{x}(k)\|^2] < \bar{C} \varepsilon^k \|\bar{x}_0\|^2$. Define K by (21) and let us show that K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$. Let $x_0 \in \mathbb{R}^n$, $r_0 \in [N]$, $s_0 \in [M]$, $\tau_0 < 0$, $\sigma_0 \in [N]$ be arbitrary. By Lemma 5.1, we can take the corresponding $\chi_0 \in \tilde{\mathcal{X}}$ such that (18) holds. Then we can see that

$$\begin{aligned} x(k+1) &= (A_{r(k)} + B_{r(k)} K_{\sigma(k), \lfloor k - \tau(k) \rfloor_T}) x(k) \\ &= (\bar{A}_{\theta(k)} + \bar{B}_{\theta(k)} \bar{K}_{\theta(k)}) x(k). \end{aligned}$$

This shows $\bar{x}(k) = x(k)$ provided the initial states of Σ and $\bar{\Sigma}$ coincide as $x_0 = \bar{x}_0$. Therefore, since $\bar{\Sigma}_{\bar{K}}$ is mean square stable, we obtain $E[\|x(k)\|^2] < C \varepsilon^k \|x_0\|^2$ for every k . Hence K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$, as desired. ■

The constraint (20) leads us to decompose $\tilde{\mathcal{X}}$ into the unobservable part $\tilde{\mathcal{X}}_h = [N] \times [M]$ and the observable part $\tilde{\mathcal{X}}_o = [N] \times [T]$ as $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_h \times \tilde{\mathcal{X}}_o$. Then the stabilization of $\bar{\Sigma}$ with the feedback control (19) satisfying the constraint (20) on feedback gains is equivalent to the stabilization of $\bar{\Sigma}$ via clustered observation [7] reviewed in Section II. Therefore, using Proposition 2.2 we immediately obtain the next theorem.

Theorem 5.3: For $R_\chi \in \mathbb{R}^{n \times n}$ ($\chi \in \tilde{\mathcal{X}}$) define $\mathcal{D}_\chi(R) = \sum_{\chi' \in \tilde{\mathcal{X}}} \bar{p}_{\chi',\chi} R_{\chi'}$. Assume that $R_\chi \in \mathbb{R}^{n \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$ satisfy the linear matrix inequality

$$\begin{bmatrix} R_\chi & A_\alpha G_{\gamma,\delta} + B_\alpha F_{\gamma,\delta} \\ G_{\gamma,\delta}^\top A_\alpha^\top + B_\alpha^\top F_{\gamma,\delta}^\top & G_{\gamma,\delta} + G_{\gamma,\delta}^\top - \mathcal{D}_\chi(R) \end{bmatrix} > 0 \quad (22)$$

for all $\chi = (\alpha, \beta, \gamma, \delta) \in \tilde{\mathcal{X}}$. For each $\gamma \in [N]$ and $\delta \in [T]$ define $K_{\gamma,\delta} = F_{\gamma,\delta} G_{\gamma,\delta}^{-1}$. Then K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$.

Example 5.4: Let $N = 3$ and consider the Markov jump linear system Σ given by the matrices

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.45 & -0.3 \\ 1.2 & 0.45 \end{bmatrix}, \quad A_2 = A_3 = \begin{bmatrix} -0.7 & 0.7 \\ 0.2 & 0.8 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \end{aligned}$$

Let the transition probabilities of the Markov state r be given by $p_{ii} = 0.6$ for every i and $p_{ij} = 0.2$ for all distinct i and j . Assume that a controller tries an observation of the Markov state every 4 time units, but it fails with probability $1/2$. The corresponding Markov chain s can be realized by letting $\tau = 4$ and $p = 1/2$ in Example 3.3. Also we set $T = 4$. Solving the linear matrix inequalities (22) we obtain stabilizing feedback gains. We construct 100 sample paths of the solution x of the stabilized system Σ_K . Figs. 3 and 4 show the sample average and the sample paths of $\|x(k)\|^2$, respectively. We can see that, even though the observation of the Markov state is not necessarily performed periodically, the designed controller attains stabilization.

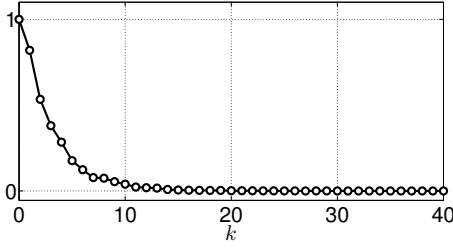


Fig. 3. Stabilized system: Sample average of $\|x(k)\|^2$

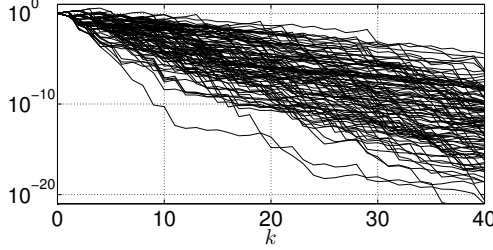


Fig. 4. Stabilized system: 100 sample paths of $\|x(k)\|^2$

A. Observation at the Initial Time

The controller designed by Theorem 5.3 is stronger than the ones in [11]–[13] in the following sense: though the controller by Theorem 5.3 does not necessarily need to know the Markov state at the initial time $k = 0$ for stabilization, the ones designed in [11]–[13] are supposed to know the Markov state at $k = 0$ for stabilization. The aim of this subsection is to show that the above two different types of stabilization are in fact equivalent provided the first observation time, t_0 , is uniformly bounded.

For a family of observation processes \mathcal{T} , we define another family of observation processes $\mathcal{T}' \subset \mathcal{T}$ by $\mathcal{T}' = \{t \in \mathcal{T} : t_0 = 0 \text{ with probability one}\}$. \mathcal{T}' expresses the set of all observation processes in \mathcal{T} that observes the Markov state at $k = 0$. In particular we can see that

$$\mathcal{T}'_{s,\Lambda} = \{t_\Lambda(s(\cdot; s_0))\}_{s_0 \in \Lambda}. \quad (23)$$

The difference from (7) is that the initial state s_0 of the Markov chain s is confined to be in Λ .

Then we can prove the following theorem.

Theorem 5.5: Assume that there exists $\tau > 0$ such that $t_0 \leq \tau$ with probability one. Then K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$ if and only if K stabilizes $(\Sigma, \mathcal{T}'_{s,\Lambda})$.

Sketch of the proof: If K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$ then K clearly stabilizes $(\Sigma, \mathcal{T}'_{s,\Lambda})$ because $\mathcal{T}'_{s,\Lambda} \subset \mathcal{T}_{s,\Lambda}$. Next assume that K stabilizes $(\Sigma, \mathcal{T}'_{s,\Lambda})$. Then, by the above observation (23), there exist $C > 0$ and $\varepsilon \in [0, 1)$ such that the solution x of Σ_K satisfies (2) for all $x_0 \in \mathbb{R}^n$, $r_0 \in [N]$, and $s_0 \in \Lambda$. Let us show that K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$. Let $x_0 \in \mathbb{R}^n$, $r_0 \in [N]$, $s_0 \in [M]$, $\sigma_0 \in [N]$, and $\tau_0 < 0$ be arbitrary. By the assumption, there exist $p_0, \dots, p_\tau \geq 0$ such that $\sum_{k=0}^\tau p_k = 1$ and $\mathcal{P}(t_0 = k_0) = p_{k_0}$. Fix a $k_0 \in \{0, \dots, \tau\}$ and consider the case $t_0 = k_0$. Let $C' = \max\{\|A_i + B_i K_{\delta, \gamma}\|\}_{i \in [N], \delta \in [N], \gamma \in [T]}$, where $\|\cdot\|$ denotes the maximum singular value of a

matrix. Then one can show $E[\|x(k_0)\|^2] \leq C'^{2k_0} \|x_0\|^2$. Since x follows the stabilized dynamics after $k = t_0$ we obtain $E[\|x(k)\|^2] \leq C \varepsilon^{k-k_0} C'^{2k_0} \|x_0\|^2$, which happens with probability p_{k_0} . Therefore, taking the summation of this inequality with respect to k_0 we can actually derive $E[\|x(k)\|^2] \leq \sum_{k_0=0}^\tau p_{k_0} C \varepsilon^{k-k_0} C'^{2k_0} \|x_0\|^2 \leq (\tau C C'^2) \varepsilon^k \|x_0\|^2$. Hence K stabilizes $(\Sigma, \mathcal{T}_{s,\Lambda})$. ■

VI. CONCLUSION

In this paper we studied the state-feedback stabilization of discrete-time Markov jump linear systems when the observation of the Markov-state by a controller is time-randomized by another Markov chain. Using an embedding of the Markov state to another Markov chain, we transformed the Markov jump linear system with time-randomized observations to the one with clustered observations. Based on this transformation we derived linear matrix inequalities for finding state-feedback stabilizing gains. The proposed method can, in a unified way, treat time-random observations including periodic and renewal-type observations studied in the literature. A numerical example is presented to show the effectiveness of the proposed method.

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